GRASP Laboratory
General Robotics & Autonomous Systems and Processes

Mechatronic Modeling and Design with Applications in Robotics

Analytical Modeling (Part 2)

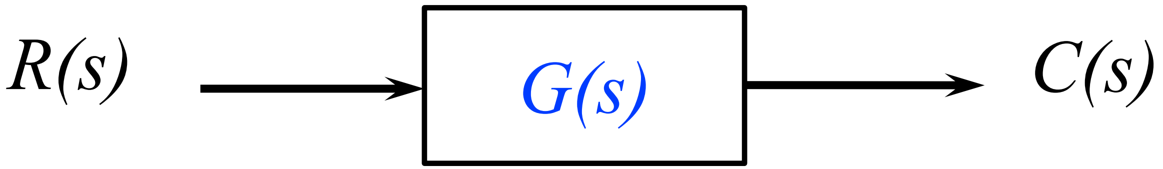
Transfer Function

- A system is assumed to be at rest (zero initial conditions),
- A transfer function is defined by

Laplace transform of **system output**

$$G(s) := \frac{C(s)}{R(s)}$$

Laplace transform of **system input**



Transfer Function

Note: input, system and output into three separate and distinct parts.

A general *n*th-order, linear, time-invariant differential equation:

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

where *c(t)* is the output, *r(t)* is the input.

Assume: zero initial conditions, and take the Laplace transform on both side:

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + a_{m-1} s^{m-1} + \cdots + a_0) R(s)$$

$$\Rightarrow \quad \frac{C(s)}{R(s)} = \frac{(b_m s^m + a_{m-1} s^{m-1} + \cdots + a_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)} \quad \Rightarrow \quad G(s) = \frac{C(s)}{R(s)}$$

$$C(s) = R(s) G(s)$$

Example

Find the transfer function represented by $\frac{dc(t)}{dt} + 2c(t) = r(t)$, and use the result to find the response $c(t)$ to a unit step input with zero initial conditions.

Laplace Transform

One of the most important math tool in the course!

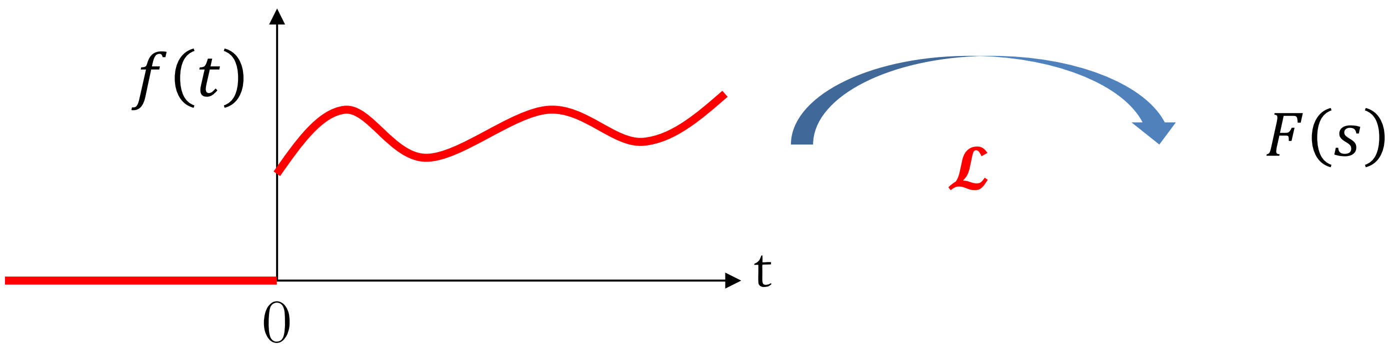
Definition:

For a function $f(t)$ ($f(t) = 0 \; for \; t = 0$)

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

(s: complex variable)

$F(s)$ is denoted as the Laplace transform of $f(t)$



Inverse Laplace Transform

Allow us to find $f(t)$ given $F(s)$:

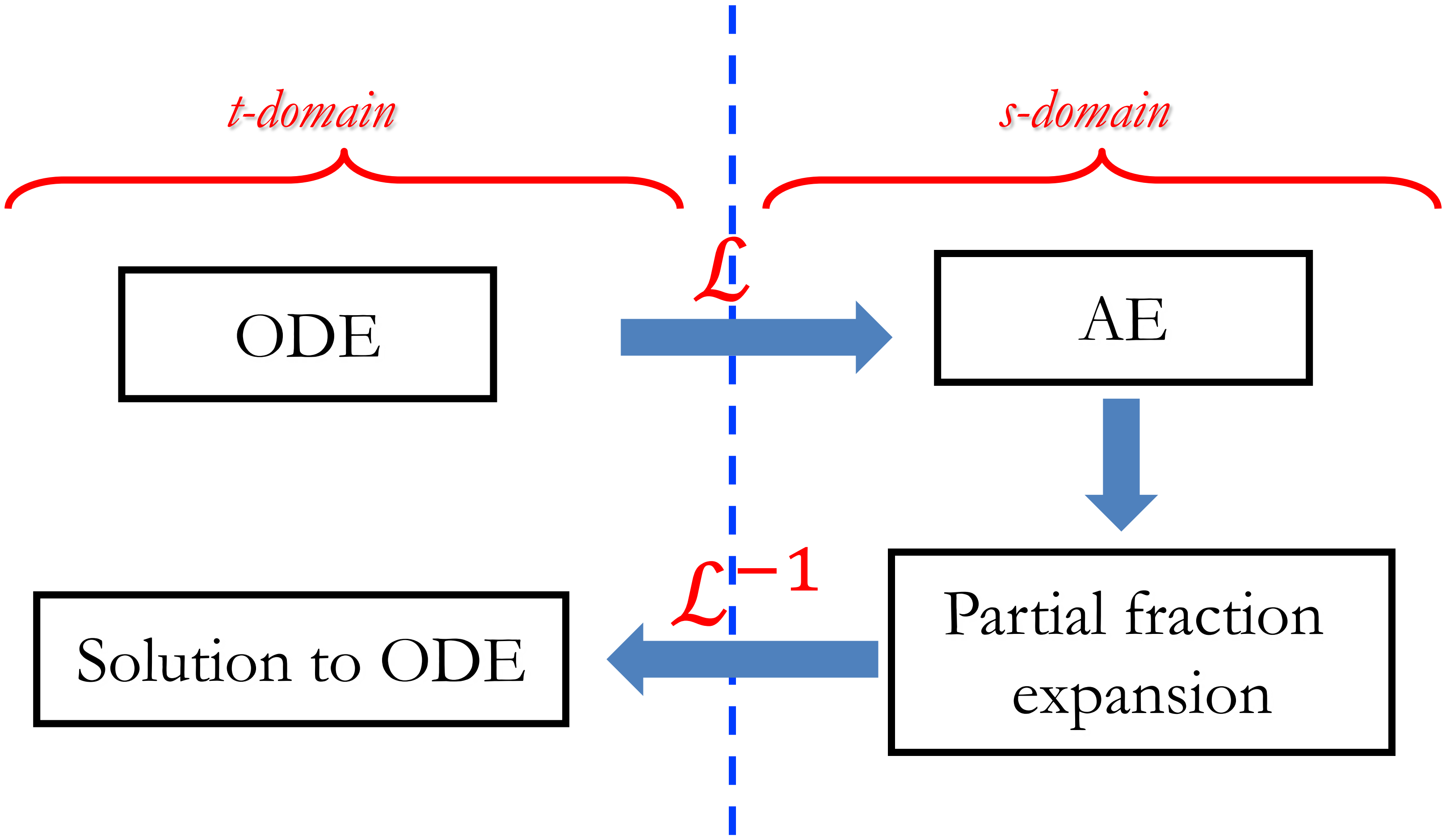
$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)$$

where

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

An Advantage of Laplace Transform

Transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Laplace Transform Table

No.	$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{-at}u(t)$	$\frac{1}{s+a}$
6	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

$\mathcal{L}$ →

← $\mathcal{L}^{-1}$

Laplace Transform Theorems (Properties)

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)\, e^{-st} dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}\left[e^{-at} f(t)\right] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT} F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - sf(0-) - f(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

Partial-Fraction Expansion

Partial-Fraction Expansion: To convert the function to a sum of simpler terms.

E.g.,

$$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

Partial-Fraction Expansion

$$F(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

Reminder:
Order of the numerator less than its denominator

$\mathcal{L}^{-1}$

$$f(t) = \mathcal{L}^{-1}\{s\} + \mathcal{L}^{-1}\{1\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2 + s + 5}\right\}$$

3 Cases (Roots of the Denominator)

1. Real and Distinct

$$F(s) = \frac{2}{(s+1)(s+2)} \rightarrow F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

2. Real and Repeated

$$F(s) = \frac{2}{(s+1)(s+2)^2} \rightarrow F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

3. Complex or Imaginary

$$F(s) = \frac{3}{s(s^2+2s+5)} \rightarrow F(s) = \frac{K_1}{s} + \frac{K_2 s+K_3}{s^2+2s+5}$$

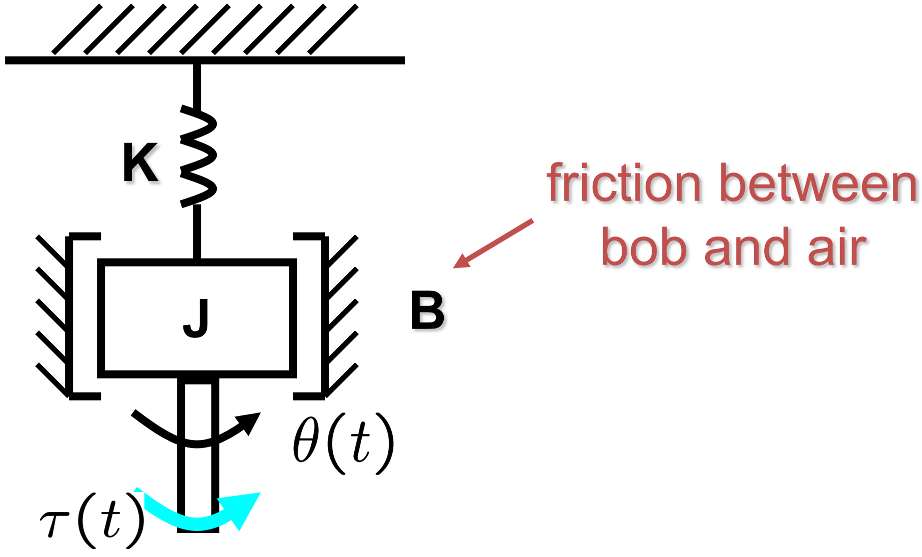
Solution to ODEs via Laplace Transform

Differentiation Theorem: $\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$; $\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0) - f'(0)$;
$\mathcal{L}\left\{\frac{d^nf}{dt^n}\right\} = s^nF(s) - \sum_{k-1}^{n} s^{n-k}\ f^{k-1}(0)$;

Example: Given the following differential equation, solve for $y(t)$ if all initial conditions are zeros.

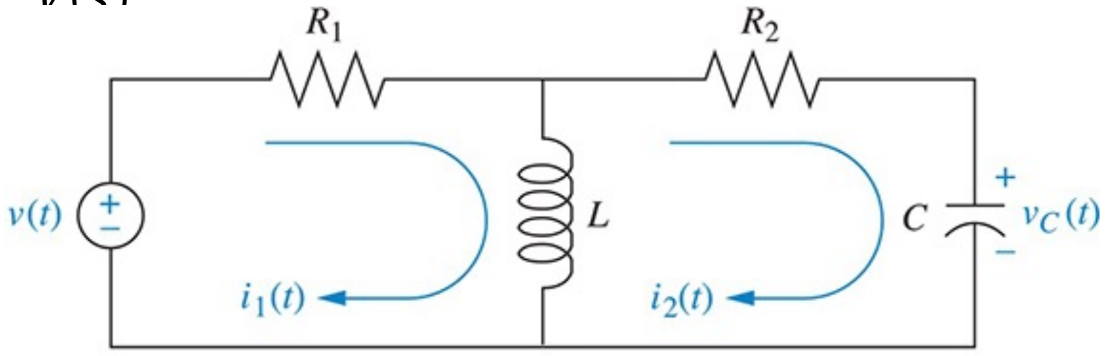
$$\frac{d^2f}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = 4u(t)$$

Example 1



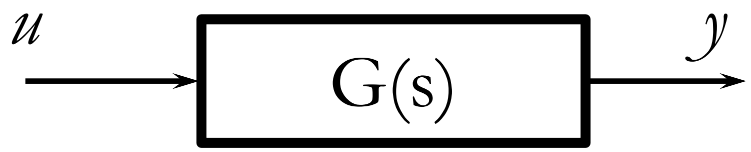
Example 2

Given the network below, find the transfer function $\frac{I_2(s)}{V(s)}$.



Converting a TF to State Space

Assume the TF of a SISO system is as follows:



$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$ where $m<n$

Then its state-space model can be written below:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \text{ where } A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}_{n\times n},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}, C = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 \end{bmatrix}_{1\times n}, D = [0]$$

Example

$$G(s) = \frac{2s^2 + 5s + 3}{3s^3 + 7s^2 - 6s + 1}$$

Please find its state-space model.

$$G(s) = \frac{\frac{2}{3}s^2 + \frac{5}{3}s + 1}{s^3 + \frac{7}{3}s^2 - 2s + \frac{1}{3}}$$ (third-order system)

Its state-space model: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{3} & 2 & -\frac{7}{3} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & \frac{5}{3} & \frac{2}{3} \end{bmatrix}, D = [0]$$

Converting from State Space to TF

Assume the state-space model of a system is as follows:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Take the Laplace Transform assuming zero initial conditions

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

Solving for X(s) in above equations

$X(s) = (sI - A)^{-1}BU(s)$ where I is the identity matrix

Substitute it to $y = Cx + Du$ ➔

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Example

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0 \cdot u$$

Please find its transfer function.

$$G(s) = C(sI - A)^{-1}B + D$$
$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + [0]$$
$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + [0]$$
$$= \frac{10s^2+30s+20}{s^3+3s^2+2s+1}$$

The End!!